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Abstract Singular Equations and Applications

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INTRODUCTION

In the previous paper [5] we considered the degenerate operational equation

$$BMu + Lu = h, \tag{1}$$

where h belongs to a complex Banach space E, B is a closed linear operator from E into itself, M, L are closed linear operators from another Banach space F into E, and $u \in D(L) \cap D(BM)$ ($D(T)$ will denote, here and in the sequel, the domain of the operator T) is the unknown function.

The degeneration in (1) arises because M has not, in general, a bounded inverse.

This abstract theory was applied to either degenerate or singular differential equations in Banach spaces. In fact, it permits one to handle some types of ordinary and partial differential equations occurring in applied mathematics.

For example, some of them have the form

$$\partial(B(t, x)\, u(t, x))/\partial t + A(t, x)\, u(t, x) = h(t, x), \quad 0 < t \leqq T, x \in \Omega,$$

where Ω is a bounded domain in R^n, and $A(t, x)$, $B(t, x)$ are suitable differential operators in $x = (x_1, ..., x_n)$ with coefficients depending on t.

In all of these examples, the operator B in (1) was therefore the operator in $L^p(0, T; X) = E$, where X is another Banach space and $p > 1$, defined by $(Bu)(t) = u'(t)$, $D(B) = W_0^{1,p}(0, T; X)$, the usual Sobolev space of order 1, with zero initial value.

As a basic hypothesis, we assumed the modified resolvent $(zM + L)^{-1}$ to exist in the sector $\operatorname{Re} z \geqq -a_0 - b_0 |\operatorname{Im} z|$, $a_0, b_0 > 0$, and the norm of $L(zM + L)^{-1}$ in $L(E)$, the space of all bounded linear operators from E into itself, to have a polynomial growth.

On the other hand, B was supposed to be an invertible closed operator

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such that $B-z$ has a bounded inverse if $\operatorname{Re} z \leqq a_1 - b_1 |\operatorname{Im} z|$, $a_1, b_1 > 0$, $b_1 > b_0$.

In the simplest case, the commutative one, in which ML^{-1} commutes with B, suitable regularity of h, in the sense that $h \in D(B^m)$ for a certain positive integer m, was sufficient to ensure both existence and uniqueness of the solution. Hence, the classical results in Gantmacher [6] were obtained as a particular case.

The non-commutative case exhibited some problems, because without further assumptions connecting B, L, and M existence and uniqueness of the solution may fail.

We then introduced a restrictive condition on the commutator $[B; L(zM+L)^{-1}]$ implying what we were looking for.

The present paper is devoted to handle Eq. (1) assuming that 0 is an isolated singularrity of some resolvents and this will make things easier, as one expects.

The technique we use in the first part of the paper is an extension of the one we developed in [3] for $B = d/dt$; but in order to handle the non-commutative case and more general ones we return to the approach in [5].

It is to be noted that J. Lagnese [8] obtained this type of result concerning singular differential equations in Hilbert space by a different technique. He also assumed that the space E could be viewed as a direct sum of two subspaces but our theory is more general in that it permits one to handle a wider range of problems.

The paper is arranged as follows. Section 1 analyzes the equation

$$MBu + Lu = h, \tag{2}$$

which may seem more natural than (1), if we recall, for example, that B often stands for d/dt, under the basic commutativity assumption that for all $z \in \rho(S)$, the resolvent set of $S = L^{-1}M$, and all $x \in D(B)$, $(S-z)^{-1}x \in D(B)$ and $(S-z)^{-1}Bx = B(S-z)^{-1}x$. In Section 2 we consider (1) under a corresponding assumption on B, L, and M. We also give some conditions ensuring that a problem of type (2) can be reduced to one of type (1).

Section 3 considers Eq. (1) assuming that 0 is a pole for $(M+z)^{-1}$ only; of course, it is clear that if we want to avoid some conditions on $(M+zL)^{-1}$ we must add other assumptions of a different kind (see [8]). In this case, we see that if L and M commute, we can develop an easy theory; we also find a general form for h ensuring that a solution of (1) exists.

In Section 4 the non-commutative case is considered and it is shown that under reasonable hypotheses on the operators entering in (1), to solve it is equivalent to solve an equation of order $k \geqq 0$ in B, with bounded operator coefficients.

The order k is connected with the order of the pole 0 of the resolvent $L(M+zL)^{-1}$. If $k \geqq 2$, the new equation may be in its turn either regular or singular.

Finally, Section 5 contains some examples of applications to various differential problems, of abstract or concrete type.

In particular, we show that the results obtained permit one to handle degenerate operational equations of second order which translate either initial value problems or boundary problems.

We also want to note that the results in Section 4 permit a general treatment of the equation $A(t)\dot{x}+x=f$ under the hypothesis of constant index for $A(t)$ (see [1]).

1. THE EQUATION $MBu+Lu=h$ IN THE COMMUTATIVE CASE

This section gives sufficient conditions to solve Eq. (2), where L, M are linear closed operators from E into F and B is a closed operator from E into itself, E, F complex Banach spaces.

To begin with and to avoid further complications, we make some hypotheses that are not the best possible ones but that explain the main points of the method we want to use. Thus we assume that M is bounded and L has a bounded inverse. Then (2) is equivalent to $L^{-1}M\,Bu+u=L^{-1}h$. Let $S=L^{-1}M$; we then shall suppose that $z=0$ is a pole of order m for the resolvent $(z-S)^{-1}$. It is well known that if ε is chosen sufficiently small, than $P=\fint_{|z|=\varepsilon}(z-S)^{-1}\,dz$, $\fint_{|z|=\varepsilon}=(2\pi i)^{-1}\int$, is a projection into $N(S^m)=N(S^n)$, $n\geqq m$, and $E=N(S^m)\oplus R(S^m)$.

Clearly, for all closed operator T, $N(T)$ denotes the null space of T and $R(T)$ its range.

The above decomposition is the key of our further work. We make the following commutativity assumption: For all $z\in\rho(S)$, the resolvent set of S, and all $x\in D(B)$, $(z-S)^{-1}x\in D(B)$ and $(z-S)^{-1}Bx=B(z-S)^{-1}x$.

Since it is an easy matter to recognize that if $u\in D(B)$ then Pu, $(1-P)\,u\in D(B)$, (2) is equivalent to

$$PSPBPu+Pu=Pf, \qquad f=L^{-1}h, \tag{3}$$

$$(1-P)\,S(1-P)\,B(1-P)\,u+(1-P)\,u=(1-P)f. \tag{4}$$

Let $(1-P)\,S(1-P)=S_2\in L(R(S^m))$; we know that S_2 has a bounded inverse and thus (4) is reduced to

$$B_2(1-P)\,u+S_2^{-1}(1-P)\,u=S_2^{-1}(1-P)f, \qquad B_2=(1-P)\,B(1-P).$$

In view of our commutativity assumption, B_2 is the restriction of B to $R(S^m)$.

This equation can be solved, for example, according to the following two methods: first, we only assume that B has a bounded inverse (and thus B_2 has this property too). Second, we apply the perturbation theory for bounded invertibility and deduce that if $\|S_2^{-1}; L(R(S^m))\|$ is small, then (4) has a unique solution for all f.

Otherwise, we assume that $B-z$ (or $B+z$) has a bounded inverse for $\operatorname{Re} z \leqq a_0 - b_0 |\operatorname{Im} z|$, where $a_0, b_0 \in R^+$ can be chosen in such a way that $\sigma(S_2^{-1})$, the spectrum of S_2^{-1}, is contained in $\operatorname{Re} z < a_0 - b_0 |\operatorname{Im} z|$.

Furthermore, there exists $C > 0$ such that $\|(B-z)^{-1}; L(E)\| \leqq C(1+|z|)^{-1}$ in the indicated sector.

It is then easy to verify that

$$(1-P)\, u = \int_{\Gamma} (S_2^{-1} + z)^{-1} (B_2 - z)^{-1} S_2^{-1} (1-P) f\, dz,$$

where Γ denotes a positively oriented circumference $|z| = r_1$ such that $|z| < r_1$ contains $\sigma(S_2^{-1})$ and is contained in $\operatorname{Re} z < a_0 - b_0 |\operatorname{Im} z|$, is the unique solution of (4).

We now turn to (3). First of all, in view of [10, p. 229], and our assumptions, $S^n P = 0$ for all $n \geqq m$.

We then observe that if $Pf \in D(B)$ and S_1 denotes the restriction of S to $N(S^m)$, then a solution Pu of (3) must verify $B^2 S_1^2 Pu = Pu - Pf + S_1 BPf$. This in turn implies that if $Pf \in D(B^{m-1})$, then $B^j S_1^j Pu = (-1)^j Pu + \sum_{s=0}^{j-1} (-1)^{j+s+1} S_1^s B^s Pf$, $j = 1, 2, \ldots, m$. Hence, $Pu = \sum_{s=0}^{m-1} (-1)^s S_1^s B^s Pf$.

One easily proves that if $Pf \in D(B^m)$, then this expression really satisfies (3). We can establish the following

THEOREM 1.1. *Under the preceding assumptions, if* $PL^{-1}h \in D(B^m)$, *where* m *is the order of the pole* $z = 0$ *for the resolvent of* $L^{-1}M$, *then* (2) *has the unique solution* $u = \sum_{j=0}^{m-1} (-1)^j S_1^j PL^{-1}h + u_1$, *where* u_1 *satisfies* (4), *with* $f = L^{-1}h$.

Remark 1.2. Analogous considerations can be made if the operator $S = L^{-1}M$ is closed and its resolvent has a pole of finite multiplicity at $z = 0$. In fact, what we established in Theorem 1.1 holds true even if M is unbounded and L is the identity operator.

If $D(L)$ is a subspace of $D(M)$, then we can also consider the problem in the following way: We see S as a bounded operator from $D(L)$ into itself. As $(S+z)^{-1} = L^{-1}(z + ML^{-1})^{-1} L$, we deduce that $z = 0$ is a pole for $(z+S)^{-1}$ if $z = 0$ is a pole for $(z + ML^{-1})^{-1}$ (and hence for the $L(F; E)$-valued function $(zL + M)^{-1}$). The equation $SBu + u = L^{-1}h$ is to be viewed in the space $D(L)$ with the graph norm and Theorem 1.1 will be applicable if the part $\tilde{B}$ of B in $D(L)$ satisfies $(zL+M)^{-1} L\tilde{B}u = \tilde{B}(zL+M)^{-1} Lu$ for all $u \in D(\tilde{B})$.

As an example let $\mathscr{L}$, $\mathscr{M}$ be two linear closed operators from the Banach space X into another Banach space Y, with $D(L) \subseteqq D(\mathscr{M})$ everywhere dense in X. Let L, M be the operators induced in $L^p(0, T; X)$ by means of $(Lu)(t) = \mathscr{L}u(t)$, $(Mv)(t) = \mathscr{M}v(t)$, $u \in L^p(0, T; D(\mathscr{L}))$, $v \in L^p(0, T; D(\mathscr{M}))$, $p \in (1, +\infty)$, $T > 0$. Then we shall consider L, M as bounded operators from $L^p(0, T; D(\mathscr{L}))$ into $L^p(0, T; Y)$. If B is the operator in $L^p(0, T; X)$ defined by $(Bu)(t) = u'(t)$, $u \in W_0^{1,p}(0, T; X)$, then $\tilde{B}$ coincides with the restriction of B to $W_0^{1,p}(0, T; D(\mathscr{L}))$ and thus the preceding commutativity assumption is satisfied.

In the sequel, when applications to abstract differential equations will be considered, for the sake of simplicity, we shall identify $\mathscr{L}$ with L and $\mathscr{M}$ with M. On the other hand, it is easy to recognize that if $z = 0$ is a pole for $(z + \mathscr{L}^{-1}\mathscr{M})^{-1}$, then this holds also for $(z + L^{-1}M)^{-1}$. Furthermore, note that the decomposition $X = N((\mathscr{L}^{-1}\mathscr{M})^m) \oplus R((\mathscr{L}^{-1}\mathscr{M})^m)$ implies the corresponding one $L^p(0, T; X) = L^p(0, T; N(S^m)) \oplus L^p(0, T; R(S^m))$.

Remark 1.3. The case in which $D(M) \subseteqq D(L)$ can be reduced to the one where $D(M) = D(L)$ if one assumes that there is $c \neq 0$ such that $L + cM$ has a bounded inverse and $B - c$ has the same properties as B. In fact it is enough to write (2) in the equivalent form $M(B - c)\, u + (L + cM)\, u = h$. We shall have then to suppose $z = 0$ to be a pole for the resolvent of $(L + cM)^{-1}M$ in a suitable space. Since, in a formal way, $((L + cM)^{-1}M + z)^{-1} = (1 + cz)^{-1}(z(1 + cz)^{-1}L + M)^{-1}(L + cM)$, if $L(zL + M)^{-1}$ has a pole with finite multiplicity at $z = 0$, and the restriction of B to $D(M)$ commutes with $(L + cM)^{-1}M$, then Theorem 1.1 applies.

2. The Equation $BMu + Lu = h$ in the Commutative Case

This section is devoted to solving (1), where L, M are closed linear operators from F into E and B is closed from E into itself. We also add the conditions that $D(L) \subseteqq D(M)$ is everywhere dense in F and that L have a bounded inverse; hence $T = ML^{-1} \in L(E)$. Under these hypotheses, Eq. (1), which appears less natural than (2), nevertheless is easier to handle than (2).

The commutativity assumption now reads: for all $u \in D(B)$, we have $Tu \in D(B)$ and $TBu = BTu$. Furthermore, we suppose that $z = 0$ is a pole of $(T + z)^{-1}$ of finite multiplicity, and hence $zL + M$ has a bounded inverse for all z, $0 < |z| \leqq \varepsilon$, where ε is sufficiently small.

As $(z + T)^{-1} = L(zL + M)^{-1}$ and L has a bounded inverse, if m is the order of the pole 0, we have $\|(zL + M)^{-1}; L(E; F)\| \leqq C\, |z|^{-m}$, $0 < |z| \leqq \varepsilon$.

Let $Lu = v$. Then (1) is reduced to $BTv + v = h$ and we can apply to this equation the considerations we did in section 1.

In this case P is the projection $\oint_{|z|=\varepsilon} L(zL+M)^{-1}$ onto $N(T^m)$. If T_1 and T_2 denote the restriction to T to $N(T^m)$ and $R(T^m)$, respectively, one then recognizes that the following result holds:

THEOREM 2.1. *Under the preceding assumptions, if* $T^jPh \in D(B^j)$ *for* $j=1,..., m-1$ (*in particular, if* $h \in D(B^{m-1})$), *then* (1) *has a unique solution* u *given by* $u=\sum_{j=0}^{m-1}(-1)^j L^{-1}B^jT_1^jPh + L^{-1}T_2^{-1}\quad S(1-P)\,h$, *where* $S(1-P)\,h$ *is the solution of* $Bw+T_2^{-1}w=(1-P)\,h \in R(T^m)$.

Remark 2.2. If $m=1$, there are no conditions on h.

Remark 2.3. If we assume that $L^{-1}M$ is a closed operator in E such that 0 is a pole for $(z+L^{-1}M)^{-1}$ and h is a little more "regular" than what we supposed in Theorem 1.1, then (2) can be reduced to a problem of type (1). In fact, if $L^{-1}h \in D(B)$ and, as we have assumed, B has a bounded inverse, then (1) is equivalent to $BSBu+Bu)=BL^{-1}h$; this is an equation of type (1). Otherwise, if $D(L) \subseteq D(M)$, we write $(z+L^{-1}M)^{-1}$ as $(zL+M)^{-1}L$ and then the hypotheses to be done are the ones in Remark 1.2; that is, $z=0$ is a pole for $L(zL+M)^{-1}$, and the part $\tilde{B}$ of B in $D(L)$ satisfies $(zL+M)^{-1}L\tilde{B}u=\tilde{B}(zL+M)^{-1}Lu$ for all $u \in D(\tilde{B})$.

Remark 2.4. In the results we have obtained it has always been assumed that L is invertible. This is an important assumption which avoids serious troubles. If L does not have this property, but there exists $c \neq 0$ such that $cM+L=L_1$ has a bounded inverse, then we could apply our theory if $(z+ML_1^{-1})^{-1}$ has a pole of order m at $z=0$, for (1) is written as $B_1M_1u+L_1u=h$, $B_1=B-c$, $M_1=M$.

On the other hand, if $z \neq 1/c$, since $zL+M=(1-zc)(z(1-zc)^{-1}L_1+M_1)$, and $(wL_1+M_1)^{-1}$ exists for all $w\colon 0<|w|\leqq\varepsilon$, we deduce that $zL+M$ is invertible provided $0<|z(1-zc)^{-1}|\leqq\varepsilon$; that is, as ε can be supposed to be small, provided $0\neq z$ in a suitable neighbourhood of 0. Therefore in any case $zL+M$ must have a bounded inverse for $0<|z|\leqq\varepsilon_1$.

Remark 2.5. We saw in [5] that under certain hypotheses on the operators entering in the problem, we can reduce the more general problem $\tilde{B}Mu+Lu=h$, $\Gamma Mu=v_0$, where $\tilde{B}$ is a closed operator from E into itself and Γ is a surjective bounded operator from $D(\Gamma)$ ($\subseteq D(\tilde{B})$) into X, X being a Banach space which can be identified with a subspace of E. The reduction of (2) to (1) which we spoke of in Remark 2.4 implies that analogous arguments can be adapted for $M\tilde{B}u+Lu=h$, $\Gamma u=u_0=\Gamma\tilde{u}$, where $\tilde{B}$ and Γ satisfy some conditions corresponding to the preceding ones.

This can, in fact, be recognized directly, as if we request that $\tilde{u}\in D(\tilde{B})\cap D(L)$, then our equation is equivalent to $MB(u-\tilde{u})+$

$L(u-\tilde{u})=h-M\tilde{B}\tilde{u}-L\tilde{u}$, which is a problem of type (2). Here B is the operator defined by $D(B)=\{u \in D(\tilde{B})\colon \Gamma u=0\}$, $Bu=\tilde{B}u$.

This remark permits one, in particular, to handle initial value problems with initial conditions $\neq 0$.

3. The Case in Which $z=0$ Is a Pole for $(z+M)^{-1}$

In the theory of degenerate differential equations the key role is played by the modified resolvent $(zM+L)^{-1}$. Now, unless M, L are the operators induced by certain matrices, the finite-dimensional case, which makes the problem of inverting $zM+L$ a quite algebric one (see [4, 6]), it may be very difficult to study directly this invertibility.

In some important concrete applications we know that $z=0$ is a pole for $(z+M)^{-1}$ (and this implies, in particular, $E=F$) and in this section we give some conditions which guarantee solvability for (1) or (2). In particular, it shall turn out that in this framework the best results are obtained when $D(M) \subseteq D(L)$. Note that we could develop a theory extending the one in [8], but we only want to give verifiable and simple conditions implying the results we are looking for.

Le us suppose that $D(M) \subseteq D(L)$ and that there exists $0 \neq c \in C$ such that $cM+L$ is a closed operator with domain $D(M)$.

Write (1) as $B_1 Mu+(cM+L)u=h_1$, $B_1=B-c$. Since $D(M)=D(cM+L)$, if B_1 satisfies the properties we requested for B in Sections 1 and 2, we have to solve a problem equivalent to (1) in which the domains of the two operators L and M coincide. Henceforth we shall assume this and thus it is not restrictive to suppose $D(L)=D(M)$. Our further hypothesis consists in that L and M commute, in the sense that $(L-z')^{-1}(M-z'')^{-1}=(M-z'')^{-1}(L-z')^{-1}$ for all $z' \in \rho(L)$ and $z'' \in \rho(M)$, and 0 is a pole of order $m \geqq 1$ for $(M+z)^{-1}$. Of course, it is to be supposed that B and M also commute in the sense we specified above. Since $E=N(M^m) \oplus R(M^m)$, if P denotes the projection onto $N(M^m)$ and u satisfies (1), then it is easy to recognize that Pu and $(1-P)u$ belong to $D(L)$ and $LPu=PLu$. Hence, (1) is equivalent to the system

$$BMPu+LPu=Ph, \tag{5}$$

$$BM(1-P)u+L(1-P)u=(1-P)h. \tag{6}$$

If M_2 denotes the restriction of M to $R(M^m)$, then M_2 has a bounded inverse and thus LM_2^{-1} is bounded from $R(M^m)$ into itself. By our hypotheses we have immediately that (6) has a unique solution for all h.

As long as (5) is concerned we first observe that if $h=LM^{m-1}f$, where $f \in D(L^m)$, then (5) has $Pu=M^{m-1}Pf$ as its solution, without any other assumption on L. Hence for any h of the above form, problem (1) has at least one solution.

Assume now L to be invertible. Then, if $Ph \in D(B^{m-1}L^{m-1})$, and hence $Ph \in D(B^k M^k L^{m-1-k})$, $k = 0, \ldots, m-1$, a solution Pu of (5) must satisfy

$$B^n M^n Pu = (-1^n L^n Pu + \sum_{K=0}^{n-1} (-1)^{n-1-k} B^k M^k L^{n-1-k} Ph, \qquad n \geqq 1.$$

And thus, if one takes $n = m$, then

$$L^m Pu = \sum_{k=0}^{m-1} (-1)^k B^k M^k L^{m-1-k} Ph. \tag{7}$$

This leads to the following:

THEOREM 3.1. *Under the preceding assumptions, if $Ph \in D(B^{m-1})$ and L has a bounded inverse, then* (1) *has at least one solution.*

Proof. We need only to show that $Pu = \sum_{k=0}^{m-1} (-1)^k B^k M^k L^{-(k+1)} Ph$ is a real solution of (5). In fact,

$$\begin{aligned} BMPu &= \sum_{k=0}^{m-1} (-1)^k B^{k+1} M^{k+1} L^{-(k+1)} Ph \\ &= \sum_{k=0}^{m-2} (-1)^k B^{k+1} M^{k+1} L^{-(k+1)} Ph \\ &= -\sum_{k=1}^{m-1} (-1)^k B^k M^k L^{-k} Ph, \\ LPu &= \sum_{k=0}^{m-1} (-1)^k B^k M^k L^{-k} Ph. \end{aligned}$$

Q.E.D.

If $D(L) \subset D(M)$, it is clear that to solve (1) is again equivalent to solving the system (5)–(6), for if $u \in D(L)$ then also Pu and $(1-P)u$ must belong to $D(L)$ in view of commutativity. Furthermore, the same hypotheses ensure that a solution of (1) has necessarily to satisfy (7) and it is easy to see that if L has a bounded inverse, the Pu we found above satisfies (5). On the contrary, (6) presents some difficulties since LM_1^{-1} is not, in general, a bounded operator in $R(M^m)$. Hence, one must assume that LM_1^{-1} has the properties ensuring that, for example, the theory in [2] can be applied.

Now these hypotheses concern the resolvent $(z + LM_1^{-1})^{-1}$, while we are interested in having conditions on M only.

This explains because the best results from the present point of view are obtained if $D(M) \subseteq D(L)$.

One could think to avoid the commutativity hypothesis on L and M by assuming that for some complex number c, $cM+L$ has a bounded inverse, for then (1) can be put in the form $BM(cM+L)^{-1}v+L(cM+L)^{-1}v=h$ and thus $M(cM+L)^{-1}$, $L(cM+L)^{-1}$ are bounded operators which commute with each other [1]. One has therefore to assume that 0 is a pole for the resolvent of $M(cM+L)^{-1}$.

On the other hand, $z+M(cM+L)^{-1}=(zc+1)(M+z(zc+1)^{-1}L)(cM+L)^{-1}$ implies that $z=0$ must be a pole for $(M+zL)^{-1}$ and thus we return to a hypothesis of the kind used in Section 2.

If we want to handle Eq. (2) under the same assumptions, we have only to observe that if $h \in D(B)$, (2) is equivalent to $BM(Bu)+L(Bu)=Bh$, and this is an equation of type (1), to which we can apply all our preceding results.

4. The General Case

In [5] the author showed that if $D(L) \subseteq D(M)$, the operator $T=ML^{-1}$ commutes with B, while $(zT+1)^{-1}=P(z)^{-1}$ is defined when $\operatorname{Re} z \geqq -b_0 - a_0 |\operatorname{Im} z|$, and has its norm in $L(E)$ bounded by a polynom $\operatorname{pol}(|z|)$, then $u = ⨍_\Gamma z^{-k}P(z)^{-1}(B-z)^{-1}B^k h\,dz$, $P(z)=zM+L$, is the unique solution of (1). Here k is a suitable non-negative integer connected with the degree of $\operatorname{pol}(|z|)$ and Γ is a contour in the complex plane coinciding with the half lines $\operatorname{Re} z = -b_0 - a_0 |\operatorname{Im} z|$ when $|z|$ is sufficiently large.

Now, if $z=0$ is a pole for $(z+T)^{-1}$ of order $m \geqq 1$, then $P(z)^{-1}=z^{-1}L^{-1}(T+z^{-1})^{-1}$ is defined for all z: $|z| \geqq \varepsilon^{-1}$ and in a neighbourhood of 0, too, and satisfies $\|LP(z)^{-1}; L(E)\| \leqq C|z|^{m-1}$. Hence, our theory in [5] applies. It is easy, in fact, to verify directly that the u this new method furnishes with $k=m$ coincides with the solution described in Theorem 2.1. We have $u = ⨍_\Gamma z^{-m}(B-z)^{-1}\ B^m\ P(z)^{-1}\,Ph\,dz + ⨍_\Gamma z^{-m}(B-z)^{-1}\ B^m P(z)^{-1}(1-P)\,h dz = [\mathrm{i}]+[\mathrm{ii}]$, where P is the usual projection onto $N(T^m)$.

As $P(z)^{-1}\,Ph = \sum_{j=0}^{m-1}(-1)^j z^j T_1^j\,Ph$, we deduce that

$$[\mathrm{i}] = \sum_{j=0}^{m-1} (-1)^j L^{-1} ⨍_{|z|=\varepsilon} z^{-m+j}(B-z)^{-1}B^m T_1^j\,Ph\,dz$$
$$= \sum_{j=0}^{m-1} (-1)^j\,L^{-1}B^j T_1^j\,Ph.$$

As T_2, the restriction of T to $R(T^m)$, is an invertible operator, we can write [ii] in the form

$$[\mathrm{ii}] = ⨍_\Gamma z^{-m}L^{-1}(B-z)^{-1}B^m T_2^{-1}(z+T_2^{-1})^{-1}(1-P)\,h\,dz.$$

We are allowed to deform Γ into a certain circumference γ: $|z|=r$ which contains the spectrum of T_2^{-1} in its interior. Hence

$$[\mathrm{ii}] = \oint_\Gamma z^{-m} L^{-1}(B-z)^{-1}(B-z+z)\, B^{m-1} T_2^{-1}(z+T_2^{-1})^{-1}(1-P)\, h\, dz$$

$$= \oint_\gamma z^{-m} L^{-1} B^{m-1} T_2^{-1}(z+T_2^{-1})^{-1}(1-P)\, h\, dz + \oint_\gamma z^{-(m-1)} L^{-1}(B-z)^{-1}$$
$$\cdot B^{m-1} T_2^{-1}(z+T_2^{-1})^{-1}(1-P)\, h\, dz.$$

The first integral vanishes if $m \geqq 1$, as $\|(z-T_2^{-1})^{-1};\ L(R(T^m))\| \leqq C(|z|-\sigma)^{-1}$, $|z|>\sigma$, and thus we can repeat the process.

Hence,

$$[\mathrm{ii}] = \oint_\gamma L^{-1} T_2^{-1}(z+T_2^{-1})^{-1}(B-z)^{-1}(1-P)\, h\, dz,$$

and the expression in Theorem 2.1 is obtained.

Our main goal in this section is to obtain some conditions entailing solvability for (1) when the operators B and T do not commute. To this end, we use the same method as in [5] and thus we seek for a solution of (1) of the same type we found in the commutative case, but in which h is substituted by a suitable f.

Of course, we shall suppose $z=0$ to be a pole of order m for $(z+T)^{-1}$. To begin with, we shall assume that f itself belongs to $D(B^{m-1})$ and hence we put our u in the form

$$u = \sum_{j=0}^{m-1} (-1)^j L^{-1} T_1^j P B^j f + \oint_\gamma L^{-1} T_2^{-1}(z+T_2^{-1})^{-1}(1-P)(B-z)^{-1} f\, dz.$$

It follows that if we suppose the commutators $[B; T^k]$, $[B; P]$, $1 \leqq k \leqq m-1$, to be well-defined and have a bounded extension to all of E, we obtain for $m \geqq 2$

$$BMu = \sum_{j=0}^{m-2} (-1)^j B T_1^{j+1} P B^j f + \oint_\gamma B(z+T_2^{-1})^{-1}(1-P)(B-z)^{-1} f\, dz$$
$$= \sum_{j=0}^{m-2} (-1)^j [B; T^{j+1}]\, P B^j f + \sum_{j=0}^{m-2} (-1)^j T^{j+1} [B; P]\, B^j f$$
$$+ \sum_{j=0}^{m-2} (-1)^j T_1^{j+1} P B^{j+1} f + B(1-P)\, B^{-1} f - \oint_\gamma z^{-1} B(zT+1)^{-1}$$
$$\cdot (1-P)(B-z)^{-1} f\, dz.$$

The last addendum, on the other hand, coincides with

$$-\oint_\gamma z^{-1}[B;(zT+1)^{-1}](1-P)(B-z)^{-1}f\,dz+\oint_\gamma z^{-1}(zT+1)^{-1}[B;P]$$

$$\cdot(B-z)^{-1}f\,dz-\oint_\gamma z^{-1}(zT+1)^{-1}(1-P)f\,dz$$

$$-\oint_\gamma (zT+1)^{-1}(1-P)(B-z)^{-1}f\,dz.$$

Now, since we can write $(zT+1)^{-1}(1-P)f=T_2^{-1}(z+T_2^{-1})(1-P)f\in R(T^m)$ and have an estimate $\|(z+T_2^{-1})^{-1};\ L(R(T^m))\|\leqq C(|z|-\sigma)^{-1}$, $|z|>\sigma$, we see that the third integral in this expression vanishes.

Hence,

$$BMu+Lu=Pf+\sum_{j=0}^{m-2}(-1)^j\,[B;T^{j+1}]\,PB^jf$$

$$+\sum_{j=0}^{m-2}(-1)^j\,T^{j+1}[B;P]\,B^jf-[B;P]\,B^{-1}f+(1-P)f$$

$$-\oint_\gamma z^{-1}[B;(zT+1)^{-1}](1-P)(B-z)^{-1}f\,dz$$

$$+\oint_\gamma z^{-1}(zT+1)^{-1}[B;P]\,(B-z)^{-1}f\,dz.$$

If we write $z^{-1}(zT+1)^{-1}[B;P](B-z)^{-1}f$ in the form $z^{-1}[B;P](B-z)^{-1}f-T(zT+1)^{-1}[B;P](B-z)^{-1}f$, we conclude that

$$BMu+Lu=f-\oint_\gamma z^{-1}[B;(zT+1)^{-1}](1-P)(B-z)^{-1}f\,dz$$

$$-\oint_\gamma T(zT+1)^{-1}[B;P](B-z)^{-1}f\,dz$$

$$+\sum_{j=0}^{m-2}(-1)^j[B;T^{j+1}P]\,B^jf.$$

If $m=1$, that is, $z=0$ is a simple pole for $(z+T)^{-1}$, the same technique we used for $m>1$ permits us to recognize that our u satisfies

$$BMu+Lu=f-\oint_\gamma z^{-1}[B;(zT+1)](1-P)(B-z)^{-1}f\,dz$$

$$-\oint_\gamma T(zT+1)^{-1}[B;P](B-z)^{-1}f\,dz.$$

Henceforth we shall assume that $\|(B-z)^{-1}; L(E)\| \leqq C(1+|z|)^{-1}$ in a sector $\operatorname{Re} z \leqq a_0 - b_0 |\operatorname{Im} z|$, $a_0, b_0 > 0$, where the constant C can be supposed sufficiently small; in most applications we have in mind this hypothesis is satisfied. This easily implies the following:

THEOREM 4.1. *Under the conditions introduced above on the operators relative to $m=1$, then problem* (1) *has one solution for all $h \in E$.*

Assume now $m=2$. Notice that also in this case problem (1) has been reduced to an equation such as $f+[B; TP]f+Qf=h$, where the powers B^j, $j=1, 2,\ldots$, do not appear, Q is a bounded operator with a small norm, and we seek for a solution f in $D(B)$. But we cannot, in general, control the norm of $[B; TP]$ in view of applying the bounded invertibility theorem. In fact, if the commutative $[B; [B; (zT+1)^{-1}]]$ and $[B; [B; P]]$ have a bounded extension to all of E, we see that Q maps $D(B)$ into itself with a small norm. What we are able to affirm is contained in the following result:

THEOREM 4.2. *Under the preceding assumptions for $m=2$, if $[B; [B; TP]]$ is well-defined and its norm in $L(E)$ is sufficiently small, then problem* (1) *has one solution for each $h \in D(B)$.*

We only need to observe that $I+Q+[B; TP]$ is expressible as $(1+[B; TP](I+Q)^{-1})(I+Q)$.

On the contrary, it is easy, as we shall show by simple examples, (see Example 4), to see that the term $[B; TP]$ may give a perturbation such that $I+[B; TP]+Q$ has no bounded inverse and the corresponding equation is solvable under certain conditions on h.

In the general case, the equation we obtain for f is

$$f+Qf+\sum_{j=0}^{m-2} (-1)^j [B; T^{j+1} P] B^j f = h.$$

Here all coefficients are bounded operators and we can reduce it to a first-order system. If $[B; T^{m-1} P]$ has a bounded inverse, the usual technique for abstract regular equations is applicable. Otherwise, we have a new degenerate problem and thus nothing can be said in advance on its solvability.

Remark 4.3. Suppose L, $\tilde{L}$ to be two closed linear operators from F into E having the same domain D everywhere dense in F. Furthermore assume that $\tilde{L}$ is invertible, that $M\tilde{L}^{-1}$ commutes with B, and that $z=0$ is a simple pole for its resolvent (or that the hypotheses in Section 3 are verified).

Then our theory applies to the equation $BMu+\tilde{L}u=f$. Hence, for all $f \in E$ there is a unique solution u of this equation, with $u=\tilde{L}^{-1}Pf+\tilde{L}^{-1}S(1-P)f$, where P is a suitable projection and S is a bounded operator from $N(P)$ into itself. It is clear that u satisfies (1) if and only if $f-(I-L\tilde{L}^{-1})[P+S(1-P)]f=h$.

It follows that if $I-L\tilde{L}^{-1}$ has a sufficiently small norm in $L(E)$, Neumann theorem can be used again and leads to a unique solution of (1).

Let us see how this condition on $I-L\tilde{L}^{-1}$ reads when $E=L^p(0, T; X)$, $1<p<+\infty$, $B=d/dt$, with $D(B)=W_0^{1,p}(0, T; X)$, $M(t)\equiv M$, $\tilde{L}(t)\equiv\tilde{L}=L(0)$, As we have $\|I-L\tilde{L}^{-1};\, L(E)\|\leqq\sigma$ if and only if

$$\int_0^T \|f(t)-L(t)\,L(0)^{-1}f(t); X\|^p\,dt \leqq (\sigma+\varepsilon)\int_0^T \|f(t); X\|^p\,dt,$$

for all $\varepsilon>0$, an assumption suitable for our purposes is the following one: the closed linear operators $L(t)$, $0\leqq t\leqq T$, have the same domain D_L everywhere dense in X and there are $C>0$, $\alpha\in(0, 1]$ such that $\|[L(t)-L(s)]\,L(r)^{-1};\, L(X)\|\leqq C\,|t-s|^\alpha$, $t, s, r\in[0, T]$.

This condition often occurs in evolution equations. In fact, as we then have

$$\|f(t)-L(t)\,L(0)^{-1}f(t); X\|\leqq C\,t^\alpha\,\|f(t); X\|,$$

if the extremum T in the interval $[0, T]$ is supposed to be small enough, we can conclude that the problem $d(Mu(t))/dt+L(t)\,u(t)=h(t)$, $0<t\leqq T$, $Mu(T)=0$, has a solution for each $h\in L^p(0, T; X)$.

5. Applications and Examples

Example 1. Suppose that A is a closed linear operator in the complex Banach space X, with a domain $D(A)$ everywhere dense in X.

Assume that δ^{-1} is an eigenvalue for A of finite multiplicity m, or, more exactly, that $z=0$ is a pole of order m for the resolvent $(z+\delta^{-1}-A)^{-1}$.

To begin with, consider the problem

$$\begin{gathered}(1-\delta A)\,u'(t)+Au(t)=h(t), \qquad 0<t\leqq T<\infty,\\ u(0)=u_0.\end{gathered} \tag{8}$$

The change of variables $u=e^{\delta^{-1}t}v$ reduces (8) to the problem

$$\begin{gathered}\delta(1-\delta A)\,v'(t)+v(t)=\delta e^{-\delta^{-1}t}h(t)=f(t), \qquad 0<t\leqq T,\\ v(0)=u_0.\end{gathered} \tag{9}$$

If $u_0=0$, one takes $E=F=L^p(0, T; X), p>1$, $M\colon L^p(0, T; D(A))\to E$, $(Mu)(t)=(1-\delta A)\,u(t)$ and can directly apply Remark 1.2, Theorem 1.1, and Theorem 3.1. Hence, if $f \in E$ and $Pf \in W_0^{m,p}(0, T; X)$, then (9), and hence (8), has a unique strict solution according to [2] or [5].

If $u_0 \neq 0$, we may find a class of admissible initial conditions writing $v(t) = w(t) + \sum_{j=0}^{m} t^j/j!\, w_j$, $w_0 = u_0$. Then (9) becomes

$$\delta(1-\delta A)\, w'(t) + w(t) = f(t) - \delta(1-\delta A) \sum_{j=0}^{m-1} t^j/j!\, w_{j+1} - \sum_{j=0}^{m} t^j/j!\, w_j, \qquad 0 < t \leqq T, \qquad w(0) = 0. \tag{10}$$

Theorem 1.1 then affirms that if $Pf \in W^{m,p}(O, T; X)$ and $Pu_0 = \sum_{i=0}^{m-1} (-1)^i \delta^i (1-\delta A)^i (Pf)^{(i)}(0) + (-1)^m \delta^m (1-\delta A)^m u$, $u \in D(A^m)$, then there is one and only one solution for (8).

Of course, as we have observed above, (8) can be attacked by means of the results in Section 3. It is enough to suppose that the function h in (8), with $u_0 = 0$, belongs to $W_0^{1,p}(0, T; X)$ to transform (8) into

$$d((1-\delta A)\, u'(t))/dt + Au'(t) = h'(t), \qquad 0 < t \leqq T,$$
$$\lim_{t \downarrow 0} (1-\delta A)\, u'(t) = 0.$$

Hence, if $Ph' \in W_0^{m-1,p}(0, T; X)$, a solution of (8) is ensured.

Theorem 3.1 immediately applies even if we are considering problems such as either

$$(1-\delta A)\, u'(t) + A_0 u(t) = h(t), \qquad 0 < t \leqq T,\ h \in L^p(0, T; X), \qquad u(0) = 0, \tag{11}$$

or

$$d((1-\delta A)\, u(t))/dt + A_0 u(t) = h(t), \qquad 0 < t \leqq T,\ h \in L^p(0, T; X), \qquad \lim_{t \downarrow 0} (1-\delta A)\, u(t) = 0, \tag{12}$$

where A_0 is another closed operator with $D(A) \subseteq D(A_0)$ and such that $cA + A_0$ is closed for same $c \neq 0$; furthermore, A and A_0 commute in the sense we used above. The existence of one solution of (12) is then ensured by the assumption $Ph \in W_0^{m-1,p}(0, T; X)$. As long as (11) is concerned, the hypothesis $h \in W_0^{1,p}(o, T; X)$ pzrmits one to reduce (11) to a problem of the type (12) and then $Ph' \in W_0^{m-1,p}(0, T; X)$ again implies the existence of a solution. Of course, in this framework, P denotes the projection of $L^p(0, T; X)$ onto $L^p(0, T; N((1-\delta A)^m))$, induced by the corresponding one of X onto $N((1-\delta A)^m)$.

For example, consider the problem

$$(1+\partial^2/\partial x^2)\,\partial u/\partial t - \partial^2 u/\partial x^2 = F(t,x), \qquad t\in(0,T],\ x\in(0,l\pi),$$

$$u(0,x)=u_0(x), \qquad x\in(0,l\pi),$$

$$u(t,0)=u(t,l\pi)=0; \qquad l \text{ a positive integer;}$$

it occurs in some important areas of applied mathematics and it has been studied by a lot of different methods (see [8] and the references therein).

In this case $m=1$ and it is easily seen that the projection P is given by

$$Pu(t,x)=2(l\pi)^{-1}\sin x\int_0^{l\pi} u(t,\xi)\sin\xi\, d\xi.$$

Our results immediately permit one to prove that if, for example, $u_0\in H_0^1(0,l\pi)\cap H^2(0,l\pi)$, then the problem has a (unique) solution provided $\sin x\int_0^{l\pi}[e^{-t}F(t,\xi)-u_0(\xi)]\sin\xi\,d\xi$ is continuously differentiable and $\int_0^{l\pi}[F(0,\xi)-u_0(\xi)]\sin\xi\,d\xi=0$.

We want to point out explicitly that no assumptions of this kind are necessary for the corresponding problem relative to the equation $\partial((1+\partial^2/\partial x^2)\,u)/\partial t-\partial^2u/\partial x^2=F$.

Finally, we note that our theory can be applied to (11) and (12) when $1-\delta A$ is a Fredholm operator having the same nullity and deficiency indices and A_0, with $D(A)=D(A_0)$ (we can also assume $D(A)\subseteq D(A_0)$, if $cA+A_0$ is closed for a certain $c\neq 0$), is another closed operator. In fact, we can consider the restrictions of $1-\delta A$ and A_0 to $D(A)$ and hence they turn out to be bounded from $D(A)$ into X.

Then it is possible to apply the theory in [9, pp. 398, 400] concerning the inverse $(1-\delta A+zA_0)^{-1}$ in a neighbourhood of 0, $z\neq 0$.

EXAMPLE 2. Let A be a closed linear operator as in Example 1, but suppose $z=0$ is a simple pole for $R(z;A)=(z+A)^{-1}$.

Furthermore, let $A(t)$, $0\leqq t\leqq T$, be a family of closed linear operators in X having the same domain $D=D(A)$. It is supposed that $A(t)\,u$, $u\in D$, is strongly continuous on $[0,T]$, that $A(t)$ has a bounded inverse for each $t\in[0,T]$, and that $\|[A(t)-A(s)]\,A(r)^{-1};\ L(X)\|\leqq C\,|t-s|^{\alpha}$, $0\leqq t,s,r\leqq T$, for some constants $C>0$, $\alpha\in(0,1]$. If $A(0)$ commutes with A, or $A(0)=A$, then, according to Theorem 4.1 and Remark 4.3, the initial value problem

$$d((1-\delta A)\,u(t))/dt+A(t)\,u(t)=h(t), \qquad 0<t\leqq T,$$

$$\lim_{t\downarrow 0}\,(1-\delta A)\,u(t)=0$$

has one solution for all $h\in L^p(0,T;X)$, provided T is sufficiently small.

In particular, assume that the family $A(t)$ satisfies the preceding assumptions and that the operator A_0 coincides with $A(t_1)$, $0 < t_1 \leqq T$, or, more generally, that A_0 has the same domain as A, and $\| 1 - AA_0^{-1}; L(X) \| \leqq \delta$, with δ sufficiently small.

Then (12) has one solution for any $h \in L^p(0, T; X)$.

EXAMPLE 3. Let $L(t)$, $M(t)$, $0 \leqq t \leqq T$, be two families of closed linear operators from the complex Banach space Y into the complex Banach space X. Define $T(t) = M(t)\, L(t)^{-1}$ and $(Tu)(t) = T(t)\, u(t)$; assume that such a T induces a bounded linear operator from $L^p(0, T; X) = E$ into itself.

For example, let $T(t)$ be strongly continuous from X into itself.

Suppose that $z = 0$ is a pole of order $m \geqq 1$ for each $(z + T(t))^{-1}$, $0 \leqq t \leqq T$, in such a way that $(z + T(t))^{-1}$ exists for $0 < |z| \leqq \varepsilon$, where ε is independent of $t \in [0, T]$; see [1] for some hypotheses of this kind. This implies that $(z + T(t))^{-1} = \sum_{n=-m}^{\infty} z^n A_n(t)$, with $A_n(t) = \oint_\gamma z^{-(n+1)} (z + T(t))^{-1}\, dz$, γ independent of t. Hence $z = 0$ is a pole of order m for $(z + T)^{-1}$.

The previous assumption is basic to applying our theory; we recall that in [5] we analyzed an important case in which the "index" changes from $t = 0$ to $t \neq 0$.

The results in Section 4 then ensure that if $m = 1$ and $T(t)$ is supposed smooth enough, then the problem

$$
\begin{aligned}
& d(M(t)\, u(t))/dt + L(t)\, u(t) = h(t), \qquad 0 < t \leqq T, \\
& \lim_{t \downarrow 0} M(t)\, u(t) = 0
\end{aligned}
\tag{13}
$$

has one solution in the strict sense.

Define now the operator K by

$$
\begin{aligned}
(Kf)(t) = & \oint_\gamma \int_0^t e^{z(t-s)} z^{-1} \{ \partial (zT(t) + 1)^{-1} / \partial t \} \, (1 - P(t)) f(s)\, ds dz \\
& + \oint_\gamma \int_0^t e^{z(t-s)}\, T(t) (zT(t) + 1)^{-1}\, P'(t) f(s)\, ds\, dz.
\end{aligned}
$$

If $m \geqq 2$, problem (13) will have at least one solution provided there is $f \in W_0^{m-1,p}(0, T; X)$ such that

$$
f(t) - (Kf)(t) + \sum_{j=0}^{m-2} (-1)^j (T(t)^{j+1} P(t)) f^{(j)}(t) = h(t), \qquad 0 < t \leqq T.
$$

If one assumes that $m = 2$, we could apply the well-known techniques of bounded invertibility provided $-K + (TP)'$ maps continuously

$W_0^{1,p}(0, T; X)$ into itself with a small norm. If $m > 2$ we need to suppose that $(T^{m-1}P)'$ has a bounded inverse.

EXAMPLE 4. Referring to the situation in Section 4 and in connection with what we saw in Example 3, assume that T is a nilpotent operator and hence $T^m = 0$ for some integer $m \geqq 2$. A solution of (1) is then sought for in the form $u = \sum_{j=0}^{m-1} (-1)^j T^j B^j f$ and one easily recognizes that this u satisfies (1) if and only if $f + \sum_{j=0}^{m-2} (-1)^j [B; T^{j+1}] B^j f = h$.

The following two trivial examples show that, in general, nothing can be said in advance on the possibility of solving the new equivalent equation, in the sense that there may be one solution either for each h or only for the h's satisfying some compatibility conditions.

Let α, β, γ: $[0, T] \to C$ be three continuously differentiable functions such that $\alpha(t)^2 = -\beta(t)\gamma(t)$, $t \in [0, T)$. Then, if $T(t)$ is defined by $T(t)(u, v) = [\alpha(t) u + \beta(t) v, \gamma(t) u - \alpha(t) v]$, we see that $z = 0$ is a pole for $(T+z)^{-1}$ of order 2 where T is the operator induced in $L^p(0, T; C^2)$, $p > 1$, by $T(t)$.

Let $h = [h_1, h_2]$, $f = [f_1, f_2]$. Then the preceding condition reads $(\alpha'(t) + 1) f_1 + \beta'(t) f_2 = h_1$, $\gamma'(t) f_1 + (1 - \alpha'(t)) f_2 = h_2$, and such a system is uniquely solvable for any h if and only if $1 - \alpha'(t)^2 - \beta'(t)\gamma'(t) \neq 0$.

Consider first the case $\alpha(t) = t$, $\beta(t) = 1$, $\gamma(t) = t^2$. Then f must satisfy $2f_1 = h_1$, $2tf_1 = h_2$, and hence we have solvability, without uniqueness, only if $h_2 = th_1$. On the other hand, if $\alpha(t) = \gamma(t) = t$, $\beta(t) = -t$, solvability is ensured because now the condition reads $2f_1 - f_2 = h_1$, $f_1 = h_2$.

EXAMPLE 5. As a further application of our results, let us consider the second-order equation

$$A_2 B_1^2 u + A_1 B_1 u + A_0 u = h, \tag{14}$$

where A_i, $i = 0, 1, 2$, is a closed linear operator from F into E, with $D(A_0) \subseteq D(A_1)$, $D(A_2)$ everywhere dense in E, and B_1 is a closed linear operator from E into itself satisfying the assumptions in Section 1 for B. Suppose furthermore that A_0 has a bounded inverse.

If we put $B_1 u = v$, (14) is equivalent to the system

$$A_0^{-1} A_1 B_1^2 u + A_0^{-1} A_2 B_1 v + u = A_0^{-1} h, \qquad -B_1 u + v = 0. \tag{15}$$

We note that now the non-homogeneous term $[A_0^{-1} h, 0]$ belongs to $D(A_0) \times D(A_0) = \mathscr{E}$. On the other hand, in view of our hypotheses on the operators A_i, we can see $A_0^{-1} A_1$, $A_0^{-1} A_2$ as elements of $L(D(A_0))$ and thus, if T denotes the operator defined by $T[x; y] = [A_0^{-1} A_1 x + A_0^{-1} A_2 y, -x]$, it is obvious that $T \in L(\mathscr{E})$.

As long as B_1 is concerned, we shall assume that if $u \in D(B_1) \cap D(A_0)$

then $B_1 u \in D(A_0)$ and $A_0^{-1} A_1 B_1 u = B_1 A_0^{-1} A_1 u$, $A_0^{-1} A_2 B_1 u = B_1 A_0^{-1} A_2 u$. Now, in a formal way,

$$(T+z)^{-1}[f, g]$$
$$= [zP(z)^{-1} A_0 f - P(z)^{-1} A_2 g,\ P(z)^{-1} A_0 f + P(z)^{-1} \cdot (zA_0 + A_1) g],$$

where $P(z) = z^2 A_0 + z A_1 + A_2$; this can suggest some types of conditions to be assigned to A_0, A_1, A_2 in order to apply our theory.

For example, we see that if $z = 0$ is a pole of order $m \geqq 1$ for $A_0 P(z)^{-1}$, then $z = 0$ is a pole of the same order for $(T+z)^{-1}$. In fact, it is enough to observe that $P(z)^{-1} A_0$, $P(z)^{-1} A_2$, and $P(z)^{-1}(zA_0 + A_1)$ belong to $L(D(A_0))$ and $\| zP(z)^{-1} A_0;\ L(D(A_0))\| \leqq C |z|^{-(m-1)}$, $\| P(z) A_2;\ L(D(A_0))\| \leqq C |z|^{-m}$, $\| P(z)^{-1}(zA_0 + A_1);\ L(D(A_0))\| \leqq C |z|^{-m}$ in a deleted neighbourhood of $z = 0$.

Thus the hypotheses in Theorem 1.1 are satisfied, since in this case the restriction T_2 of T to $R(T^m)$ has a bounded inverse. As

$$P[A_0^{-1} h, 0] = \int_{|z| = \varepsilon} [zP(-z)^{-1} h,\ -P(-z)^{-1} h]\, dz,$$

we must assume that $P(z)^{-1} h \in D(B_1^m)$. If $E = F$ and B_1 commutes with the operators A_i, it is sufficient to suppose that h itself belongs to $D(B_1^m)$.

Remark 5.1. If $A_1 = 0$ and $D(A_0) \subseteq D(A_2)$, since $A_0 P(z)^{-1} = (z^2 + A_2 A_0^{-1})^{-1}$, if $z = 0$ is a pole of order $p \geqq 1$ for $(z + A_2 A_0^{-1})^{-1}$, then we could apply the preceding result with $m = 2p$.

Remark 5.2. Suppose that A_1 has a bounded inverse, too, with $D(A_0) = D(A_1) \subseteq D(A_2)$ and that $z = 0$ is a simple pole for $(z + A_2 A_1^{-1})^{-1}$; hence, $\| A_1(zA_1 + A_2)^{-1};\ L(F)\| \leqq \tilde{C} |z|^{-1}$, $0 < |z| \leqq \varepsilon$. In a formal way,

$$(z^2 A_0 + z A_1 + A_2)^{-1} = A_1^{-1}(z + A_2 A_1^{-1})^{-1}(1 + z^2 A_0 A_1^{-1}(z + A_2 A_1^{-1})^{-1})^{-1},$$

and hence

$$A_0 P(z)^{-1} = A_0 A_1^{-1}(z + A_2 A_1^{-1})^{-1}(1 + z^2 A_0 A_1^{-1}(z + A_2 A_1^{-1})^{-1})^{-1}.$$

Therefore, if $0 < |z| \leqq \min\{2\tilde{C} \| A_0 A_1^{-1};\ L(E)\|)^{-1}, \varepsilon\}$, then

$$|z|^2 \| A_0 A_1^{-1};\ L(E)\|\ \| (z + A_2 A_1^{-1})^{-1};\ L(E)\| \leqq \tfrac{1}{2}.$$

On the grounds of this estimate we deduce that $1 + z^2 A_0 A_1^{-1}(z + A_2 A_1^{-1})^{-1}$ has a bounded inverse and that $\| A_0 P(z)^{-1};\ L(F)\| \leqq C |z|^{-1}$. Hence our theory applies with $m = 1$.

Referring to Example 1, we can take $1+\partial^2/\partial x^2$, with zero boundary conditions, as A_2 and two linear differential operators with constant coefficients as A_0 and A_1.

Remark 5.3. Assume A_1 is a "small" perturbation of A_0 in the sense that $A_1 = A_0 + \tilde{A}_1$, with $\|\tilde{A}_1;\ L(D(A_0); F)\| = \eta$. Suppose $z=0$ is a simple pole for $(z+A_2A_0^{-1})^{-1} = A_0(zA_0+A_2)^{-1}$, $0<|z|\leqq \varepsilon$, and thus $\|(z+A_2A_0^{-1})^{-1};\ L(F)\| \leqq C\,|z|^{-1}$ on this domain.

This implies that if $0<|z|\leqq \delta = (-1+\sqrt{1+4\varepsilon})/2$, then $\|(z(z+1)+A_2A_0^{-1})^{-1};\ L(F)\| \leqq C\,|z|^{-1}\,|z+1|^{-1}$ and also $\|z\tilde{A}_1A_0^{-1}(z(z+1)+A_2A_0^{-1})^{-1};\ L(F)\| \leqq C\,|z+1|^{-1}\,\|\tilde{A}_1A_0^{-1};\ L(F)\| \leqq C\eta(1-\delta)^{-1}$. If $\eta \leqq (1-\delta)(2C)^{-1}$, then $1+z\tilde{A}_1A_0^{-1}(z(z+1)+A_2A_0^{-1})^{-1}$ has a bounded inverse for all $0<|z|\leqq\delta$ and as $z^2A_0+zA_1+A_2 = (1+z\tilde{A}_1A_0^{-1}(z(z+1)+A_2A_0^{-1})^{-1})(z(z+1)+A_2A_0^{-1})\,A_0$, we deduce that for these values of z, $P(z)$ has a bounded inverse satisfying $\|A_0P(z)^{-1};\ L(F)\| \leqq C\,|z|^{-1}$.

Hence the general theory is appliable to this situation with $m=1$.

As a last example we want to consider an elliptic problem.

EXAMPLE 6. Consider the problem

$$M(d^2/dt^2+c)\,u(t)+Lu(t)=h(t), \qquad 0<t<1,$$
$$u(0)=u(1)=0,$$

where L, M are closed linear operators from the Banach space X into itself, L has a bounded inverse, and $D(L)\subseteq D(M)$. We know that if the constant c is suitably chosen, then the operator B defined in $L^p(0,T;X)=E$ by $D(B)=\{u\in E;\ u',\ u''\in E,\ u(0)=u(1)=0\}$, $(Bu)(t)=u''(t)+cu(t)$, satisfies the conditions in Section 1.

In order to apply the results we obtained in that section we only need to assume that $z=0$ is a pole of order $m\geqq 1$ for $L(zL+M)^{-1}$ as an $L(X)$-valued function, and, furthermore, that the inverse of the restriction T_2 of $T=L^{-1}M$ to $R(T^m)$, where T is to be viewed as a bounded operator from $D(L)$ into itself, has the spectral properties that permit one to apply both the results in [2, pp. 371–377] and those in [7, p. 197].

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